10.2) Series:

* Definition:

- In section 10.1, we defined a sequence as a function that produces a set of number:

$$a_n = a_1, a_2, a_3, a_4, \ldots, a_n$$

- In this section, we define a series as the sum of those numbers:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n$$

* Notation:

- an infinite series is denoted by:

$$\sum_{n=K}^{n=\infty} a_n$$

- $n = K$ represents the first term
- $n = \infty$ represents the infinite term
- Σ represents the sum of terms between

1st term and the nth term.

* n^{th} partial sum:

. Let $\{a_n\}$ be a sequence of numbers:

. $a_n = a_1, a_2, a_3, \ldots, a_n$ Then the sum:

$S_1 = a_1$ represents the first term

$S_2 = a_1 + a_2$ represents the sum of the first two terms of the given series

$\vdots$

$$S_n = a_1 + a_2 + a_3 + a_4 + \ldots, a_n = \sum_{n=1}^{n} a_n$$

* Geometric Series:

. are serieses of the form:

$$a + ar + ar^2 + ar^3 + \ldots ar^{n-1} = \sum_{n=1}^{\infty} ar^{n-1}$$

. where a and r are 2 constant numbers:

. r: is the common ratio, the common ratio represent

how much is the each term changing.

* Geometric sum and values of r:

. The Geometric sum is given by:

$$\left\{ S_n = a_0 \frac{1-r^n}{1-r} \right\}$$

. where a_0 is the 1st term and r the common ratio.

(refer for proof of this formula on my other file)

. values of r:

- Theorem: if $|r| < 1$, then the series converges

and sum is:

$$\left\{ S_n = \frac{a_0}{1-r} \right\}$$

- Theorem: if $|r| \geq 1$, then the series diverges

and sum is:

$$\left\{ S_n = \infty \right\}_{\text{if } r>1} \text{ and } \left\{ S_n = n a_0 \right\}_{\text{if } r=1}$$

. Also refer to proof.

- ex: $\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^{n-1}$

- since $r = \frac{1}{2}$ which is < 1, we conclude the series converges, and n^{th} partial sum is:

$$S = \frac{3}{1-\frac{1}{2}} = \frac{3}{\frac{1}{2}} = 6$$

* Telescoping Series:

- The telescoping series is a special type of series known for its "alternating" and "cancelling" pattern:

$$\left\{ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \right\}$$

- by performing partial fraction decomposition:

$$\left\{ \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \right\}$$

then:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) \cdots \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

- we would be left with:

$$1 - \frac{1}{n+1}$$ that is n^{th} partial sum

- we now see that:

$$\lim_{n \to \infty} S_n = 1 - \underbrace{\cancel{\frac{1}{n+1}}}_{0} = \boxed{1}$$

- thus the telescoping series converges and its sum is 1

* n^{th} term Divergence Test:

- The nth term test is the first out of many tests in determining whether a series converges or diverges.

- the nth term divergence test states that:

$$\left\{ \text{if } \lim_{n \to \infty} a_n \neq 0 \text{ then the series diverges} \right\}$$

- try not to get confused between convergence and divergence in sequence and series

- we then also conclude that:

$$\left\{ \text{if } \sum a_n \text{ converges} \xrightarrow{\text{then}} \lim_{n \to \infty} a_n = 0 \right\}$$

- Note: However if $\lim_{n \to \infty} a_n = 0$ then we dont know if $\sum a_n$ converges or diverges

- ex: $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges by integral test eventhough $\lim_{n \to \infty} \frac{1}{n} = \lim_{n \to \infty} \frac{1}{n^2} = 0$

- Refer to proofs

* Combining Series :

- if $\sum a_n = A$ and $\sum B_n = B$ then:

i) $\sum (a_n + B_n) = \sum a_n + \sum B_n = A + B$

ii) $\sum (a_n - B_n) = \sum a_n - \sum B_n = A - B$

iii) $\sum K \cdot a_n = K \sum a_n = KA$ where K is a ct

* Other Properties:

i) Every non 0 ct multiplied with a divergent series would still give a divergent series

ii) if $\sum a_n$ converges and $\sum B_n$ diverges, then:

$\left\{ \sum (a_n + B_n) \text{ and } \sum (a_n - B_n) \text{ both Diverge} \right\}$

* Reindexing :

- we can rewrite $\sum_{n=1}^{\infty} a r^{n-1}$ as $\sum_{n=0}^{\infty} a r^n$

by altering the 1st term which automatically changes n^{th} term (from r^{n-1} to r^n) while keeping the number of terms the same